Optical excitable waves

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We show that an optical system, a laser with an injected signal, behaves as an excitable medium. This property gives rise to propagating pulses if the active medium is spatially extended. We study the properties of those waves and we demonstrate that they may annihilate or cross each other depending on the control parameter values. [S1063-651X(98)02503-3]

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Excitable waves have been studied for a long time in biological and chemical systems which have been considered as the only ones to present such phenomena. From a mathematical point of view, excitable waves were described by reaction-diffusion equations [1]. A typical feature of these waves is, for example, that they annihilate each other during the process of crossing. Only recently have experimental and theoretical work done on liquid crystals [2,3] shown the existence of excitable waves in an almost purely mechanical system. In the latter case the waves appeared as spiral ones very similar to those observed in the Belousov-Zhabotinsky reaction [4]. In a recent work, Argentina *et al.* [5] studied the annihilation or survival of excitable pulses after crossing. The transition from one behavior to another was associated to a global bifurcation of the nucleation solution associated with the ignition process (ignition solution).

The aim of this paper is to give an example of excitable waves in an explicit optical system: the laser with an injected signal [7]. This system allows the variation and control of the parameters large enough to observe both processes: annihilation and crossing of pulses. In the first part we show that the usual model of a laser with an injected signal, including diffraction effects can be reduced to a single one partial differential equation for the phase of the electromagnetic field in some special region of parameter space. Excitable pulses are then studied analytically in the framework of this equation. Finally the problem of the annihilation or the crossing of pulses after their collision is considered.

We consider a laser with an injected signal. This dynamical system is described in terms of the well-known Maxwell-Bloch equations for a collection of two level atoms in the slowly varying approximation and including diffraction effects:

$$\partial_t E = -\kappa[(1+i\theta)E - P - iaE_{xx}] + i\delta E + f,$$
$$\partial_t P = -\gamma_\perp[(1-i\theta)P - ED] + i\delta P, \qquad (1)$$
$$\partial_t D = -\gamma_\parallel\left(D - \Lambda + \frac{1}{2}(E^*P + EP^*)\right),$$

where E, P, and D are the electromagnetic field, the atomic polarization and the population inversion, respectively, κ, $\gamma_\perp$, and $\gamma_\parallel$ are their respective loss rates, Λ is the pumping rate, f is the amplitude of the external electric field, and a is the diffraction coefficient. δ is the detuning of the external signal with respect to the laser frequency without external field $\omega_R = (\kappa\omega_C + \gamma_\perp\omega_A)/(\kappa + \gamma_\perp)$ and θ is the detuning between the cavity frequency and the atomic frequency normalized to the losses $\theta = (\omega_C - \omega_A)/(\kappa + \gamma_\perp)$. Equation (1) considers only one spatial variable x perpendicular to the direction of the propagation of the electromagnetic field. A more general case can be treated by adding another spatial coordinate, however, one-dimensional system is enough for our purposes, and several lasers may work properly in this condition as with semiconductor lasers which possess a large dimension in one transverse spatial direction while the other is smaller than the wavelength. The spatial dependence on the direction of propagation z was eliminated by considering valid the uniform field approximation and a single longitudinal mode laser, which are common assumptions in laser theory. Furthermore, the system is considered to have a very large aspect ratio such that the boundaries will not affect its dynamical behavior. It is worthwhile to note that this model, even if too simple to describe many real lasers with circular aperture or/and multimode operation, it may apply for broad area edge emitter semiconductor lasers. We consider that the pump and the external field are independent of the transverse coordinate x which is also a common assumption in laser theory, and that it does not represent a restriction to experimental setups. However, the aim of this paper is to report the possibility of observing excitable waves in optics, and we hope it will encourage further work on the subject, both theoretically and experimentally. Our model has stationary solutions corresponding to the frequency locking to the the external field. We first demonstrate that, for some parameter range, the stable locked state is excitable in the sense that a finite excitation leads to pulse propagation. Close to the laser threshold Eq. (1) can be reduced to a modified Ginzburg-Landau equation [8]. A small parameter is defined as $\Lambda \equiv \Lambda_c + \epsilon^2$, where Λ_c represents the critical pump parameter. One assumes that δ and f are small quantities which scale as $\delta \equiv \epsilon^2\tilde{\delta}$ and $f \equiv \epsilon^3\tilde{f}$, where $\tilde{\delta}$ and $\tilde{f}$ are order unity. One then introduces "slow" time and space variables $\epsilon^2 t \equiv T$ and $\epsilon x \equiv X$. This analysis is valid closed to the laser threshold. However, for very small ϵ, and therefore very close to threshold, the correlation length will diverge and the aspect ratio of the system is small. Thus, excitable waves are no longer possible. As we increase the pump values, the correlation length decreases and the boundary conditions becomes less important because the system increases its aspect ratio. It must be noted that the results of the numerical simulations do not depend on particular boundary conditions provided the aspect ratio is big enough to support excitable waves.

The fields E, P, and D are then expanded in a series of ϵ. At the first order one gets

$$E_1 = A(X,T), \quad P_1 = (1+i\theta)A(X,T), \quad D_1 = 0.$$

At the second order one gets

$$E_2 = P_2 = 0, \quad D_2 = (1-|A(X,T)|^2.$$

The equation to be satisfied by $A(X,T)$ is found as a solvability condition at the third order:

$$A_T = (\Gamma_r + i\Gamma_i)A(1-|A|^2) + (D_r + iD_i)A_{XX} + i\tilde{\delta}A + (\gamma_r + i\gamma_i), \tag{2}$$

where Γ_r, Γ_i, D_r, and D_i have been given in [8] and

$$\gamma_r = \frac{D_i}{a\kappa}\tilde{f}, \quad \gamma_i = -\frac{D_r}{a\kappa}\tilde{f}.$$

Let us introduce the real amplitude and the phase of the forcing $\gamma \exp(i\Theta_0) \equiv \gamma_r + i\gamma_i$. The angle Θ_0 represents the relative phase of the locked laser and the external field when $\tilde{\delta}=0$. By simple algebra we find

$$\tan(\Theta_0) = \frac{-D_r}{D_i} = \frac{2\kappa^2\gamma_\perp \theta}{\theta^2(\kappa-\gamma_\perp) - (\kappa+\gamma_\perp)}.$$

Therefore Θ_0 is determined by the ratio between diffusion and diffraction in the optical system. Notice that for vanishing detuning, the phase Θ_0 vanishes, and the locked state is in phase with the external signal. In Eq. (2), all the coefficients have magnitudes which can vary from zero to unity. In order to obtain a simple interpretation of the excitability of the locked state, let us consider an asymptotic limit of Eq. (2) in which one assumes $\gamma \ll 1$. New time and space variables are then introduced as $\eta T = \tau$ and $\sqrt{\eta}X = \xi$, where η scales the smallness of γ and $\tilde{\delta}$: $\gamma = \eta\gamma'$ and $\tilde{\delta} = \eta\delta'$, where γ' and δ' represent finite quantities. At the order η^0 one gets

$$A(1-|A|^2) = 0,$$

whose solution is $A = \exp\{i[\Theta(\xi,\tau) - \Theta_0]\}$. At first order one obtains the equation for the phase of the electric field

$$\Theta_\tau = \delta' - \gamma' \sin(\Theta) + D_r\Theta_{\xi\xi} - D_i\Theta_\xi^2. \tag{3}$$

An obvious scaling transforms this equation into

$$\Theta_t = \Delta - \sin(\Theta) + \Theta_{xx} - \Upsilon\Theta_x^2, \tag{4}$$

where $\Delta = \delta'/\gamma'$ represents the external detuning normalized to the strength of the injected signal. When $|\Delta| < 1$ a locked solution exists and it is stable $[\Theta_L = \arcsin(\Delta)]$. The coefficient $\Upsilon = D_i/D_r$ measures the dispersive-diffusive character of the medium. When $\theta = \theta^* = \sqrt{(\kappa+\gamma_\perp)/(\kappa-\gamma_\perp)}$, the medium is purely diffusive $(\Upsilon = 0)$. Pulse solutions can be obtained analytically in the limit of a weak detuning and a weak dispersion. When $\Delta = \Upsilon = 0$, Eq. (4) reduces to the overdamped sine-Gordon equation

$$\Theta_t = -\sin(\Theta) + \Theta_{xx}, \tag{5}$$

which possesses a particular stationary kink solution

$$\Theta_{K_\pm} = \pm 4 \arctan[\exp(x)].$$

These phase kinks appear as pulses when one looks to physical quantities such as, for example, the intensity of the light, since they are 2π periodic functions of θ which can also depend on the spatial gradient of the phase $\partial_x\Theta$ and its derivatives with respect to x. For small detuning and dispersion, one can look for traveling pulses of the form

$$\Theta(x,t) = \Theta_K(x-ct) + \cdots,$$

where the ellipses represent higher order corrections. Rewriting Eq. (4) in a reference frame moving at velocity c, and multiplying it by the gradient of $\Theta_{K_\pm}$, we can integrate to obtain the speed of the propagating pulses. At the leading order in the perturbation, one obtains the velocity of the excitable pulse

$$c = \frac{\pi}{4}(\Delta - 2\Upsilon).$$

This ends the analysis which demonstrates the existence of excitable pulse solutions of the Maxwell-Bloch equation describing a laser with an injected signal. Of course, as usual, the robust solutions found using asymptotic techniques are likely to transform into more nonlinear solutions when one moves away from the limiting case. It is the case of our excitable phase pulses which becomes more amplitude and phase dependant when the forcing increases as we will show later numerically. We now address the question of the ignition: how do we characterize the excitations which lead to the generation of pulses? This question is related to the nucleation theory of the first order phase transition. It was addressed in general for excitable media in [6]. We first show the existence of the ignition solution and then sketch the study of its stability. The ignition solution is obtained as a stationary solution of Eq. (4) which possesses the property $\Theta(\infty) = \Theta(-\infty) = \Theta_L$. In other words, it is a homoclinic solution of the second order differential equation which describes the stationary solution

$$\Theta_x = \Omega,$$

$$\Omega_x = -\Delta + \sin(\Theta) + \Upsilon\Omega^2. \tag{6}$$

The locked solution corresponds to a saddle node of this differential equation. The existence of a homoclinic solution can be demonstrated for small Υ. When $\Upsilon = 0$, Eq. (6) has the simple mechanical interpretation of the Newton equation describing the dynamics of a mass unity in a potential $W(\theta) = \Delta\Theta + \cos(\Theta)$. The maxima of this potential are

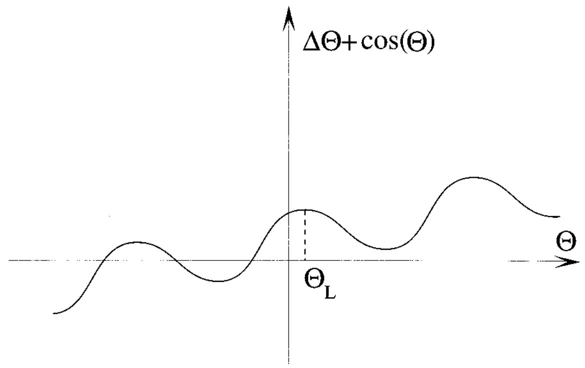

FIG. 1. Sketch of the potential $W(\theta)=\Delta\Theta+\cos(\Theta)$. The ignition solution corresponds to a trajectory that leaves the top of a hill Θ_L on the right and then bounces back to return to Θ_L.

(modulo 2π) the locked solutions (see Fig. 1). The existence of a homoclinic trajectory is then obvious.

When Υ is small, a perturbation analysis allows us to show the persistence of a homoclinic trajectory. In fact, for small Υ the Melnikov condition, which insures the existence of the homoclinic solution reads $\int H_x^3 dx=0$ where $H(x)$ is the homoclinic solution of the unperturbed system, and it is always satisfied since H_x is an odd function. The reversibility property of Eq. (6), $\Theta\rightarrow\Theta,\Omega\rightarrow-\Omega,x\rightarrow-x$ is at the origin of the property of persistence. The stability of the ignition solution $\Theta_I(x)$ can be obtained analytically for $\Upsilon=0$ and Δ close to 1. More generally, for $\Upsilon=0$, and arbitrary $|\Delta|<1$, the linearized equation of the perturbation $\Theta(x,t)=\Theta_I(x)+w(x)\exp(\sigma t)$ is a Schrödinger equation:

$$\sigma w=-\cos[\Theta_I(x)]w(x)+w_{xx}. \tag{7}$$

Thanks to the space translational invariance of Eq. (4), $d\Theta_I/dx$ is a solution of this equation corresponding to a zero eigenvalue. Since this eigenfunction has a single node, a general property of the Schrödinger spectrum [9], allows us to assert that Eq. (7) possesses only one solution with a positive eigenvalue. The stable manifold of the ignition solution W_s (see Fig. 2) thus acts as a separatrix in the infinite dimensional phase space of Eq. (4). On one side of this manifold, the flow describes a smooth convergence towards the locked state. On the other side it describes the process of the nucleation of two pulses which then move apart from one another and leads eventually to the locked state when they disappear at infinity or though the boundary. We denote the part of the one-dimensional unstable manifold which corresponds to the nucleation of two pulses from the ignition solution as W_u^+. This picture is obviously valid for small values of Υ since the ignition solution is hyperbolic.

As expected excitable pulses do exist for parameter values where Eq. (4) or Eq. (2) are no longer good approximations of Eq. (1). In this case the analysis is more difficult and can only be done numerically. We sketch here only the basic steps. We first locate parameter values where the homogeneous locked solution E_L exists and is stable. We then restricts the parameter values to those where this is the only stable homogeneous solution of Eq. (1). We restrict furthermore the parameter values in order to insure the existence of

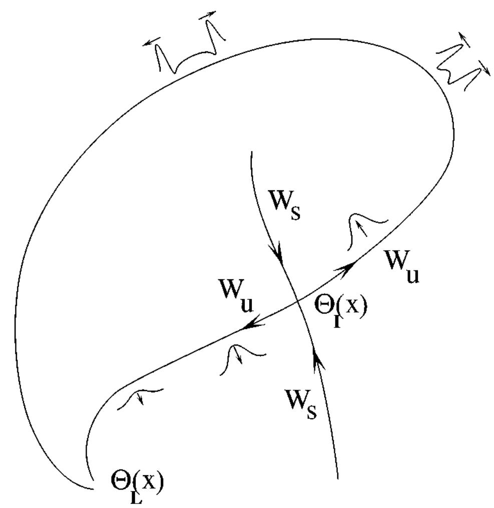

FIG. 2. Sketch of local phase portrait close to the ignition solution in the functional space of the phase equation.

an ignition solution. This solution $E_I(x)$ is then an homoclinic solution of the fourth order differential equation which describes the stationary solution of Eq. (1). The results are presented in Fig. 3(a), where we show the annihilation of two pulses after their collision.

In a slightly different parameter regime, the excitable pulses cross after collision [see Fig. 3(b)]. This surprising behavior was described in [5,6]. It is related to an Andronov homoclinic bifurcation [10] of the ignition solution. At the bifurcation point, the collision of two pulses leads to the exact ignition solution. The signature of this change of behavior can be traced back to an Andronov homoclinic bifurcation of the space independent solution of the Maxwell-Bloch equations which leads to a global stable limit cycle, while the excitable homogeneous locked state still exists. For the parameter values where the pulses crosses, the space independent Maxwell-Bloch equations experience bistability between this limit cycle and the excitable homogeneous state.

In conclusion, we have shown that an explicit model used in optics to describe a laser with an injected signal in an extended medium is able to support excitable waves. Fur-

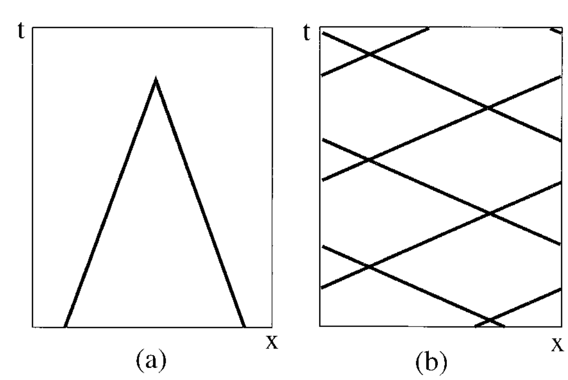

FIG. 3. Collision experiments ($\Lambda=5$, $a=1$, $\delta=-0.99$, $\theta=0.92$, $\gamma_\parallel=1,\gamma_\perp=50,\kappa=3$). (a) Annihilation of the pulses ($f=1.85$).(b) Crossing of the pulses ($f=1.70$).

thermore those excitable waves may annihilate or cross each other depending on the values of the control parameters. Analyzing several models of other related optical problems, such as bistable passive systems, a laser with optical feedback [11], or a laser with a saturable absorber [12], it is easy to recognize that they possess in principle the necessary elements to show the kind of phenomena we described above.

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